拡大同相写像とカントール集合となる極小集合について (一般および幾何学的トポロジーにおける諸問題と応用)

Kato, Hisao

数理解析研究所講究録 1303 巻 2003 年 68-72

拡大同相写像とカントール集合となる極小集合について

加藤久男 (Hisao Kato)

筑波大学数学系 (Institute of Mathematics, University of Tsukuba)

1 Introduction.

All spaces considerd in this paper are assumed to be metric spaces. *Maps* are continuous functions. By a *compactum* we mean a nonempty compact metric space. A *continuum* is a connected nondegenerate compactum. A homeomorphism $f: X \to X$ of a compactum X with metric d is called *expansive* (see[4], [12] and [13]) if there is $c>0$ such that for any $x, y \in X$ and $x \neq y$, then there is an integer $n \in \mathbf{Z}=\{0, \pm 1, \pm 2, .., \}$ such that

$$d(f^{n}(x), f^{n}(y))>c.$$

A homeomorphism $f: X \to X$ of a compactum X is *continuum-wise expansive* [5] if there is $c>0$ such that if A is a nondegenerate subcontinuum of X, then there is an integer $n \in \mathbf{Z}$ such that

$$\operatorname{diam} f^{n}(A)>c,$$

where $\operatorname{diam} B=\sup\{d(x, y) |\, x, y \in B\}$ for a set B. Such a positive number c is called an *expansive constant* for f. Note that each expansive homeomorphism is continuum-wise expansive, but the converse assertion is not true. There are many continuum-wise expansive homeomorphisms which are not expansive (eg., see [5], [6] and [8]). By the definitions, we see that expansiveness and continuum-wise expansiveness do not depend on the choice of the metric d of X. These notions have been extensively studied in the area of topological dynamics, ergodic theory and continuum theory (see [1]-[13]).

In [11], R. Mañé proved that minimal sets of expansive homeomorphisms are 0-dimensional. More generally, minimal sets of continuum-wise expansive homeomorphisms are 0-dimensional (see [5]). Also, for each continuum-wise expansive homeomorphism $f: X \to X$ of a compactum X with $\dim X>0$, there is an f-invariant closed subset Y of X such that $\dim Y>0$ and $f|Y: Y \to Y$ is weakly chaotic in the sense of Devaney (see [9]). In this paper, we prove the following result: If $f: X \to X$ is a continuum-wise expansive homeomorphism of a compactum X with $\dim X=1$, then there is a Cantor set Z in X such that for some natural number N, $f^{N}(Z)=Z$ and $f^{N}|Z: Z \to Z$ is semiconjugate to the shift homeomorphism $\widetilde{\sigma}: \widetilde{\Sigma} \to \widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is the Cantor set $\{0,1\}^{\mathbf{Z}}$. As a corollary, there is a family $\{C_{\alpha} |\, \alpha \in \Lambda\}$ of minimal sets C_{α} of f such that each C_{α} is a Cantor set, $\mathrm{Cl}\left(\bigcup\{C_{\alpha} |\, \alpha \in \Lambda\}\right)=Y$ is 1-dimensional and $f|Y: Y \to Y$ is weakly chaotic in the sense of Devaney. Also, we study infinite minimal sets of continuum-wise fully expansive homeomorphisms.

2 Continuum-wise expansive homeomorphisms and infinite minimal sets.

Let X be a compact metric space with metric d and $C(X)$ the hyperspace of all nonempty subcontinua of X with the Hausdorff metric d_H defined by

$$d_H(A,B) = \inf\{\epsilon > 0 \,|\, B \subset N(A,\epsilon), A \subset N(B,\epsilon)\ \}$$

for closed nonempty subsets A, B of X, where $N(A,\epsilon)$ denotes the ϵ-neighborhood of A in X. Let $f : X \to X$ be a homeomorphism. A nonempty closed subset M of X is a *minimal set* of f if M is f-invariant, i.e., $f(M) = M$, and no proper nonempty closed subset A of M is f-invariant. Note that a closed subset M of X is a minimal set of f if and only if for any $x \in M$,

$$\begin{gathered} M = \omega(x) = \{y \in X \,|\, \text{there is a sequence} \\ \mathrm{n}_1 < n_2 < \ldots, \text{of natural numbers such that } \lim_{i\to\infty} f^{n_i}(x) = y\}. \end{gathered}$$

Note that every homeomorphism of a compactum has a minimal set. If a minimal set M is a finite set, then M is a periodic orbit of some point $x \in X$, i.e., $M = \{x\ (= f^n(x)), f(x), f^2(x), .., f^{n-1}(x)\}$. If a minimal set M is an infinite set, then M is perfect. If an infinite minimal set M is 0-dimensional, then M is a Cantor set.

Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$. Note that every minimal set of f is 0-dimensional (see [5, Theorem (5.2)]). Consider the following sets (see [9]):

(1) $\mathcal{I}(f) = \{A \,|\, A$ is an f-invariant closed subset of $X\}$.

(2) $\mathcal{M}_\infty(f)$ is the set of all infinite minimal sets of f. If $M \in \mathcal{M}_\infty(f)$, then M is a Cantor set.

(3) $\mathcal{I}^+(f) = \{A \in \mathcal{I}(f) \,|\, \dim A > 0\}$.

(4) $\mathcal{D}(f)$ is the set of all minimal elements in the partial order of $\mathcal{I}^+(f)$ by inclusion. Note that $\mathcal{D}(f) \neq \phi$ (see [9]).

Let Σ be the Cantor set, i.e., $\Sigma = \{0,1\}^\omega$. The shift map $\sigma : \Sigma \to \Sigma$ is defined by $\sigma(x_0, x_1, x_2, ...,) = (x_1, x_2, ...,)$ for each $(x_0, x_1, x_2, ...,) \in \Sigma$. Also, let $\widetilde{\Sigma} = \{0,1\}^{\mathbf{Z}} (= \{(x_n)_n | \, x_n \in \{0,1\}, n \in \mathbf{Z}\})$. The shift homeomorphism $\widetilde{\sigma} : \widetilde{\Sigma} \to \widetilde{\Sigma}$ is defined by $\widetilde{\sigma}((x_n)_n) = (x_{n+1})_n$ for $(x_n)_n \in \widetilde{\Sigma}$. Note that $\widetilde{\Sigma}$ is identified with the inverse limit of the inverse sequence $\{\Sigma, \sigma\}$ and $\widetilde{\sigma}$ is the homeomorphism induced by σ. Let $q : \widetilde{\Sigma} \to \Sigma$ be the natural projection. Then $q \cdot \widetilde{\sigma} = \sigma \cdot q$.

First, we obtain the following theorem.

(2.1) Theorem. *If $f : X \to X$ is a continuum-wise expansive homeomorphism of a compactum X with* $\dim X = 1$, *then there is a Cantor set Z in X such that for* *some natural number N, Z is f^N-invariant and $f^N|Z : Z \to Z$ is semiconjugate to the shift homeomorphism $\widetilde{\sigma} : \widetilde{\Sigma} \to \widetilde{\Sigma}$, i.e., there is an onto map $p : Z \to \widetilde{\Sigma}$ such that $\widetilde{\sigma} \cdot p = p \cdot (f^N|Z)$.*

For the proof of (2.1), we need the followings.

(2.2) Lemma (see [5, (2.4) and (2.5)]). *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$. Let $c > 0$ be an expansive constant of f and $0 < 2\epsilon \leq c$. Then there is a positive number $\delta \leq \epsilon$ satisfying the following conditions:*

(1) $\mathbf{V}^s(\delta;\epsilon) \neq \phi$ or $\mathbf{V}^u(\delta;\epsilon) \neq \phi$, where

$$\mathbf{V}^s(\delta;\epsilon) = \{A \in C(X) |\, \operatorname{diam} A \geq \delta, \textit{ and } \operatorname{diam} f^n(A) \leq \epsilon \textit{ for each } n \geq 0\},$$
$$\mathbf{V}^u(\delta;\epsilon) = \{A \in C(X) |\, \operatorname{diam} A \geq \delta, \textit{ and } \operatorname{diam} f^{-n}(A) \leq \epsilon \textit{ for each } n \geq 0\}.$$

In particular, if $A \in \mathbf{V}^s(\delta;\epsilon)$, then $\lim_{n\to\infty} \operatorname{diam} f^n(A) = 0$. If $A \in \mathbf{V}^u(\delta;\epsilon)$, then $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0$.

(2) For each $\gamma > 0$ there is a natural number $N = N(\gamma)$ such that if A is a subcontinuum of X with $\operatorname{diam} A \geq \gamma$, then either $\operatorname{diam} f^n(A) \geq \delta$ for all $n \geq N$ or $\operatorname{diam} f^{-n}(A) \geq \delta$ for all $n \geq N$.

(2.3) Lemma. *Let X be a 1-dimensional compactum. For any $\epsilon > 0$ there is a family $\{U_1, U_2, .., U_m\}$ of open subsets of X such that $\mathrm{Cl}(U_i) \bigcap \mathrm{Cl}(U_j) = \phi\, (i \neq j)$, $\operatorname{diam} U_i < \epsilon$ for each i and the diameters of components of $X - (\bigcup_{i=1}^m U_i)$ are less than ϵ.*

To prove the next theorem (2.5), we need the following lemma.

(2.4) Lemma. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$ and $Y \in \mathcal{D}(f)$. Let c, ϵ and δ be positive numbers as in (2.2). Then for each $0 < \gamma \leq \delta$ and nonempty open subset V of Y there is a natural number $J = J(V,\gamma)$ such that if $A \subset Y$ and $A \in \mathbf{V}^u(\gamma;\epsilon)$, then there is a natural number $j = j(A)$ such that $1 \leq j \leq J$ and $f^j(A) \bigcap V \neq \phi$.*

(2.5) Theorem. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X = 1$ and $Y \in \mathcal{D}(f)$. Then there is a sequence $M_1, M_2, ..,$ of minimal sets of $f|Y$ such that each M_n is a Cantor set and $\lim_{n\to\infty} d_H(Y, M_n) = 0$. In particular,*

$$\mathrm{Cl}(\bigcup\{M |\, M \in \mathcal{M}_\infty(f|Y)\}) = Y.$$

(2.6) Proposition. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$ and $Y \in \mathcal{D}(f)$. Then there is a sequence $M_1, M_2, ..,$ of minimal sets of $f|Y$ such that $\lim_{n\to\infty} d_H(Y, M_n) = 0$.*

Let $f : X \to X$ be a homeomorphism of a compactum X. Then f is *sensitive* if there is $c > 0$ such that if $x \in X$ and U is any neighborhood of x in X, there is $y \in U$ and a natural number $n \geq 1$ such that $d(f^n(x), f^n(y)) > c$. f is *topologically transitive* if there is a point $x \in X$ such that the orbit $\{x, f(x), f^2(x), .., \}$ of x is dense in X. Also, f is *weakly chaotic in the sense of Devaney* (see [9]) if f is sensitive, f is topologically transitive and $\mathrm{Cl}(\bigcup\{M|\ M \text{ is a minimal set of } f\ \}) = X$.

(2.7) Remark. In (2.5), we see that $f|Y : Y \to Y$ is weakly chaotic in the sense of Devaney (see [9]).

3 Infinite minimal sets of continuum-wise fully expansive homeomorphisms.

A homeomorphism $f : X \to X$ of a continuum X is *continuum-wise fully expansive* provided that for any $\epsilon > 0$ and $\delta > 0$, there is a natural number $N = N(\epsilon, \delta) > 0$ such that if A is a subcontinuum of X and $\operatorname{diam} A \geq \delta$, then either $d_H(f^n(A), X) < \epsilon$ for all $n \geq N$, or $d_H(f^{-n}(A), X) < \epsilon$ for all $n \geq N$. By the similar proofs as before, we obtain the following result.

(3.1) Theorem. *Let $f : X \to X$ be a continuum-wise fully expansive homeomorphism of a nondegenerate continuum X. Then*

(1) there is a Cantor set Z in X such that for some natural number N, Z is f^N-invariant and $f^N|Z : Z \to Z$ is semiconjugate to the shift homeomorphism $\widetilde{\sigma} : \widetilde{\Sigma} \to \widetilde{\Sigma}$, and

(2) there is a sequence $M_1, M_2, ..,$ of minimal sets of f such that each M_n is a Cantor set and $\lim_{n\to\infty} d_H(X, M_n) = 0$. In particular,

$$\mathrm{Cl}(\bigcup\{M|\, M \in \mathcal{M}_\infty(f)\}) = X.$$

(3.2) Example. Let $f : T^2 \to T^2$ be an Anosov diffeomorphism, say

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

on the 2-dimensional torus T^2. Then f is a continuum-wise fully expansive homeomorphism. Hence $\mathrm{Cl}(\bigcup\{M|\, M \in \mathcal{M}_\infty(f)\}) = T^2$.

(3.3) Problem. Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X \geq 2$. In this case, are the conclusions of (2.1) and (2.5) true ?

References

[1] N. Aoki, Topological dynamics, in: Topics in general topology (eds. K. Morita and J. Nagata), *Elsevier Science Publishers B. V.* (1989), 625-740.

[2] B. F. Bryant, Unstable self-homeomorphisms of a compact space, *Thesis, Vanderbilt University, Nashville, TN* (1954).

[3] R. Devaney, An Introduction to Chaotic Dynamical Systems, 2nd ed., *Addison-Wesley*, 1989.

[4] W. Gottschalk and G. Hedlund, Topological dynamics, *Amer. Math. Soc. Colloq.,* 34 (1955).

[5] H. Kato, Continuum-wise expansive homeomorphisms, *Canad. J. Math.*, 45 (1993), 576-598.

[6] H. Kato, Chaotic continua of (continuum-wise) expansive homeomorphisms and chaos in the sense of Li and Yorke, *Fund. Math.*, 145 (1994), 261-279.

[7] H. Kato, The nonexistence of expansive homeomorphisms of chainable continua, *Fund. Math.,* 149 (1996), 119-126.

[8] H. Kato, On indecomposability and composants of chaotic continua, *Fund. Math.* 150 (1996), 245-253.

[9] H. Kato, Minimal sets and chaos in the sense of Devaney on continuum-wise expansive homeomorphisms, *Lecture Notes in Pure and Applied Mathematics*, 170 (1995), 265-274.

[10] K. Kuratowski, Topology Vol.1, *Academic Press*, Warszawa (1966).

[11] R. Mañé, Expansive homeomorphisms and topological dimension, *Trans. Amer. Math. Soc.*, 252 (1979), 313-319.

[12] W. Utz, Unstable homeomorphisms, *Proc. Amer. Math. Soc.*, 1 (1950), 769-774.

[13] R. F. Williams, A note on unstable homeomorphisms, *Proc. Amer. Math. Soc.*, 6 (1955), 308-309.

Hisao Kato
Institute of Mathematics
University of Tsukuba
Ibaraki, 305 Japan
e-mail: hisakato@sakura.cc.tsukuba.ac.jp